Solution Sheet 10

Exercise 1. Consider the congruence subgroup

$$\Gamma(3) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) \;\middle|\; \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \bmod 3 \right\}$$

and its action on $\mathbb{H}$ by Möbius transformations. Show that the quotient $\mathbb{H}/\Gamma(3)$ is a Riemann surface and that $\pi_1(\mathbb{H}/\Gamma(3)) \cong \Gamma(3)$.

Solution 1. We need to check that $z \in \mathcal{F}$ has trivial stabilizer in $\Gamma(3)$ (recall: $|\Re(z)| \leq \frac{1}{2}$ and $|z| \leq 1$). If $\frac{az+b}{cz+d} = z$, then $|cz+d|^2 = 1$. Since $|\Re(z)|^2 \leq \frac{1}{4}$, looking at $\Im(cz+d)$, we get $3c^2 \leq 4$. Now $c \equiv 0 \bmod 3$, so $c = 0$. Then, looking at $\Re(cz+d)$, $d^2 \leq 1$ and since $d \equiv 1 \bmod 3$, we obtain $d = 1$, hence $z = az + b$. Since $\Im(z) > 0$ and $a, b \in \mathbb{Z}$, this is only possible if $a = 1$ and $b = 0$ i.e. $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) = I_2$. This shows that $\Gamma(3)$ acts freely on $\mathbb{H}$. Furthermore, $\Gamma(3) \subseteq \Gamma$ acts properly and freely on $\mathbb{H}$ by biholomorphisms. So the quotient can be endowed with a structure of Riemann surface and the quotient map is a covering. The covering $\mathbb{H} \to \mathbb{H}/\Gamma(3)$ is Galois and since $\mathbb{H}$ is simply connected, we obtain $\pi_1(\mathbb{H}/\Gamma(3)) \cong \Gamma(3)$.

Exercise 2. Let $f : X \to Y$ be a covering map between connected spaces and suppose that Y is a Riemann surface. Show that X can be endowed with a (unique) Riemann surface structure such that f is holomorphic and that all covering transformations of f are biholomorphisms.

Solution 2. Let $\{(V_\alpha, \phi_\alpha)\}$ be an atlas on Y, with each V_α evenly covered by f. This means that

$$f^{-1}(V_\alpha) = \bigsqcup_i U_{\alpha,i}, \qquad f|_{U_{\alpha,i}} : U_{\alpha,i} \xrightarrow{\sim} V_\alpha.$$

We define charts on X by

$$\psi_{\alpha,i} := \phi_\alpha \circ f|_{U_{\alpha,i}} : U_{\alpha,i} \to \phi_\alpha(V_\alpha) \subset \mathbb{C}.$$

These cover X. If $(U_{\alpha,i}, \psi_{\alpha,i})$ and $(U_{\beta,j}, \psi_{\beta,j})$ overlap, then on $U_{\alpha,i} \cap U_{\beta,j}$ we have

$$\psi_{\beta,j} \circ \psi_{\alpha,i}^{-1} = \phi_\beta \circ \phi_\alpha^{-1}.$$

These functions are holomorphic since they are built from transition maps on Y. Therefore they define a Riemann surface structure on X.
Let $\tau : X \to X$ be a deck transformation. For an evenly covered subset $V_\alpha \subset Y$ and a sheet $U_{\alpha,i} \subset X$, the map τ permutes the sheets, that is, $\tau(U_{\alpha,i}) = U_{\alpha,j}$. In the charts $\psi_{\alpha,i}$ and $\psi_{\alpha,j}$ we have

$$\psi_{\alpha,j} \circ \tau \circ \psi_{\alpha,i}^{-1} = \phi_\alpha \circ f \circ \tau \circ (f|_{U_{\alpha,i}})^{-1} \circ \phi_\alpha^{-1} = \phi_\alpha \circ f \circ (f|_{U_{\alpha,i}})^{-1} \circ \phi_\alpha^{-1} = \mathrm{id}.$$

Hence τ is holomorphic. The same holds for τ^{-1}, and we conclude that τ is a biholomorphism.

Exercise 3. Show that every connected n-sheeted covering of the punctured disc $\mathbb{D}^*$ is isomorphic as a covering to

$$p_n : \mathbb{D}^* \to \mathbb{D}^*, \quad z \mapsto z^n.$$

Solution 3. We have $\pi_1(\mathbb{D}^*) \cong \mathbb{Z}$. By the classification of coverings, every connected n-sheeted covering of $\mathbb{D}^*$ is classified by the subgroup $n\mathbb{Z} \subset \pi_1(\mathbb{D}^*)$. Now consider the n-sheeted covering

$$p_n : \mathbb{D}^* \to \mathbb{D}^*, \quad z \mapsto z^n$$

The induced map $(p_n)_* : \pi_1(\mathbb{D}^*) \to \pi_1(\mathbb{D}^*)$ is multiplication by n and therefore

$$(p_n)_*(\pi_1(\mathbb{D}^*)) = n\mathbb{Z}.$$

Hence by the classification, a n-sheeted covering $q : Y \to \mathbb{D}^*$ is isomorphic to p_n.

Exercise 4. (for credit, due on 30 November) *(5 points)*
Let G be a finite group. Show that there exists a compact Riemann surface X, a finite set of points $B \subset \mathbb{P}^1$, and a holomorphic map $f : X \to \mathbb{P}^1$ with branch locus B such that the deck group of the restriction

$$f|_{X \setminus f^{-1}(B)} : X \setminus f^{-1}(B) \to \mathbb{P}^1 \setminus B$$

is isomorphic to G. **Hint:** Let G be generated by $g_1, \ldots, g_r$. Realize G as a quotient of $\pi_1(\mathbb{P}^1 \setminus \{r + 1 \text{ points}\})$.

Solution 4. Let G be generated by the finitely many elements $g_1, \ldots, g_r$. Let F_r be the free group on generators $x_1, \ldots, x_r$. The homomorphism

$$\phi : F_r \to G, \quad x_i \mapsto g_i$$

is surjective by construction. Let $Y = \mathbb{P}^1 \setminus \{a_1, \ldots, a_{r+1}\}$ be the $r + 1$ punctured sphere. From Sheet 7, Exercise 4 we know that $\pi_1(Y) \cong F_r$. By composing this isomorphism with ϕ we obtain a surjective homomorphism $\psi : \pi_1(Y) \to G$. Let $N := \ker \psi \subset \pi_1(Y)$. The subgroup N corresponds by the classification of covering spaces theory to a $|G|$-sheeted covering $p : X^\circ \to Y$ with $N = p_*(\pi_1(X^\circ))$. Since N is normal, this covering is Galois and therefore

$$\mathrm{Deck}(p) \cong \pi_1(Y)/N \cong G.$$

Since p is a local homeomorphism, X° can be endowed with a unique Riemann surface structure such that p is a holomorphic covering map.
We now extend this cover over the punctures. For each puncture a_j, choose a small disc $D_j \subset \mathbb{P}^1$ around a_j containing no other punctures, and set $D_j^* := D_j \setminus \{a_j\}$. Each connected component W of $p^{-1}(D_j^*)$ is a connected e_W-sheeted cover of the punctured disc (for some $e_W \geq 1$), and hence (by the Exercise 3) is isomorphic to

$$\mathbb{D}^* \to \mathbb{D}^*, \quad w \longmapsto w^{e_W}.$$

This extends holomorphically across $w = 0$. Doing this for all j yields a finite-sheeted covering $f : X \to \mathbb{P}^1$ that extends p. Because the fibers of f are finite and $\mathbb{P}^1$ is compact, this implies that X is compact as well. Every deck transformation of p extends holomorphically over the added points, hence gives a deck transformation of f. Conversely, any deck transformation of f restricts to one of p. Thus we conclude

$$\mathrm{Deck}(f) \cong \mathrm{Deck}(p) \cong G.$$

Exercise 5. Let $f : X \to Y$ be a covering map between compact Riemann surfaces. Show that f is a Galois covering if and only if the field extension $\mathcal{M}(X)/\mathcal{M}(Y)$ is Galois. In this case $\mathrm{Gal}(\mathcal{M}(X)/\mathcal{M}(Y)) \cong \mathrm{Deck}(f)$.